Exercise 17

The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

Solution

For a one-dimensional rod, the density is the ratio of mass to length.

$$\lambda = \frac{\Delta m}{\Delta x}$$

If the mass varies continuously, then the limit needs to be taken as $\Delta x \to 0$.

$$\begin{aligned}\lambda(x) = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} &= \frac{dm}{dx} \\ &= \frac{d}{dx}(3x^2) \\ &= 6x\end{aligned}$$

If $x = 1$ m, then the density (in kilograms per meter) is

$$\lambda(1) = 6(1) = 6.$$

If $x = 2$ m, then the density (in kilograms per meter) is

$$\lambda(2) = 6(2) = 12.$$

If $x = 3$ m, then the density (in kilograms per meter) is

$$\lambda(3) = 6(3) = 18.$$

Notice that the density increases linearly with x, so the density will be highest at the right end of the rod and lowest at the left end of the rod.